Discrete-time constrained PI-like output feedback tracking controllers - A robust positive invariance and bilinear programming approach ⋆

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Abstract: This paper presents the discrete-time counterpart of a PI-like output feedback tracking controller recently proposed for linear continuous-time systems subject to state and input constraints. The considered approach exploits the internal model principle to design via robust positive invariance and the Extended Farkas' Lemma, a constrained set-point tracking controller. The polyhedral setup allows a bilinear optimization design problem that simultaneously determines the sets of admissible set-points and the plant's initial conditions. The proposal guarantees asymptotic offset-free set-point tracking, local stability, and constraint fulfillment. Simulation results show the design's effectiveness.

Keywords: Constrained Set-Point Tracking, PI-Control, Robust Positive Invariance, Discrete-time, Bilinear Programming.

1. INTRODUCTION

The output feedback tracking control problem is a significant problem in automation and control systems engineering. We can find examples of applications in the chemical and robotic industries and the development of intelligent transportation systems and self-driving cars. The Internal Model Principle (IMP) (Chen, 2014) represents a seminal result for reference tracking problems. IMP defines the conditions under which a stabilizing controller also ensures tracking. Based on the IMP, different constrained and unconstrained tracking controllers have been developed in the literature from diverse points of view. Existing approaches range from Proportional-Integral (PI) controllers (Carvalho and Rodrigues, 2019) to Model Predictive Control (MPC), and Reference and Command Governor solutions (Limon et al., 2005; Ferramosca et al., 2011; Di Cairano and Borrelli, 2015; Garone et al., 2017).

Among all the existing regulators, PI-like controllers are undoubtedly the most used in the industry to solve set-point tracking problems (Åström et al., 2006). The success of PI controllers relies upon their simplicity and, compared to MPC solutions, their minimal computational footprint for online implementation. Most existing design strategies concentrate on the constraint-free scenario, where the IMP has global validity. However, when interested in using PI controllers in constrained setups, one must apply the IMP carefully because its validity is restricted to the region where the constraints are inactive. Consequently, the design method should consider constraints to ensure that the resulting control strategy is viable (Hennet, 1995). To safeguard both constraints fulfillment and closed-loop stability, classical solutions leverage Lyapunov stability theory and the concepts of positive invariance set and contractivity, see, e.g., Castelan and Hennet (1993); Hennet (1995); Blanchini (1999); Tarbouriech et al. (2011); Blanchini and Miani (2015); Dorea (2009); Dantas et al. (2018).

Concerning PI-control design methods for constrained control systems, existing approaches vary according to the nature of the considered plant's model. Solutions based on Algebraic Riccati Equations (ARE) or Linear Matrix Inequalities (LMI) have been developed for linear systems (Tarbouriech et al., 2000; Flores et al., 2008); Fuzzy Lyapunov functions have been used for nonlinear systems represented by Takagi-Sugeno (TS) models (Lopes et al., 2020); Lyapunov functions have been leveraged for Linear Parameter-Varying (LPV) systems (Figueiredo et al., 2020); a "quasi"-LMI approach has been designed for periodic reference signals and uncertain linear systems (Flores et al., 2009). Commonly, such solutions only deal with symmetrical input saturation constraints and define contractive ellipsoidal (or composite ellipsoidal) invariant regions. As exceptions to the mentioned literature are the proposals of Martins et al. (2020) and Santos et al. (2021), which leverage polyhedral invariant sets and bilinear optimization to design constrained PI-controllers for monovariable discrete-time and continuous-time systems, respectively.

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In addition, Santos et al. (2023) recently proposed a novel approach for designing PI-like for linear continuous-time systems subject to state and input constraints. Differently from the cited literature, in Santos et al. (2023), the authors exploit the concept of robust positive invariance to design a locally stabilizing PI-like tracking controller with a feedforward term capable of dealing with asymmetrical polyhedral state and input constraints. In particular, they used algebraic robust positive invariance relations to define a bilinear optimization design problem capable of simultaneously computing the PI controller parameters, the set of admissible reference signals, and the polyhedral state space region where the controller is robust positively invariant (Blanchini and Miani, 2015).

The present work proposes the discrete-time counterpart of the results developed recently for continuous-time systems. Thus, it shares the following features with Santos et al. (2023): *(i)* they are not limited to plants having a single input and single output structure; *(ii)* the state, input, and reference constraints can be asymmetric; *(iii)* the magnitude of the integral error can be taken into account in the design and treated as a design parameter; and *(iv)* the respective constrained set-point tracking control problem formulates as a bilinear programming problem. Moreover, compared to our proposal, the discrete-time design of Martins et al. (2020) does not consider *(i)-(iii)*.

It is worth noting that the theoretical results and optimization-based design proposed in the present work closely follow the ones in Santos et al. (2023). Thus, the current work's simple but useful contribution consists in the design of a constrained PI controller directly in discrete-time.

The rest of the paper is organized as follows. Section 2 recalls basic definitions and formulates the constrained tracking design problem. Section 3 proposes the constrained PI-like controller's design, and Section 4 shows a numerical example that contrasts the discrete and continuous-time solutions. Finally, Section 5 concludes the paper.

2. BASIC RESULTS AND PROBLEM STATEMENT

This section first recalls basic definitions for polyhedral sets, Extended Farkas' Lemma, and robust positively invariant sets. Then, it presents the considered setup and states the constrained set-point tracking problem.

2.1 Preliminaries

The following definitions are adapted from Hennet (1995); Blanchini and Miani (2015).

***Definition** 1. (Convex Polyedral Set) Any closed and convex polyhedral set $\mathcal{P}(\phi) \subseteq \mathbb{R}^n$ can be characterized by a shaping matrix $P \in \mathbb{R}^{l_p \times n}$ and a vector $\phi \in \mathbb{R}^{l_p}$, with l_p and n being positive integers, i.e.,*

$$\mathcal{P}(\phi) = \{x \in \mathbb{R}^n : Px \leq \phi\}. \tag{1}$$

Note that $\mathcal{P}(\phi)$ in (1) includes the origin as an interior point if $\phi > 0$. In the sequel, if $\phi = \mathbf{1}_* = [1, 1, \dots, 1]^T \in \mathbb{R}^*$, the resulting polyhedral set $\mathcal{P}(\mathbf{1}_*)$ will be simply denoted as $\mathcal{P}$.

***Definition** 2. (Non-negative Matrix) A matrix M is non-negative, if $M_{ij} \geq 0, \forall i$ and j.*

***Lemma** 1. (Extended Farkas' Lemma) Consider two polyhedral sets of $\mathbb{R}^n$ defined by $\mathcal{P}_i(\phi_i) = \{x \in \mathbb{R}^n, P_i x \leq \phi_i\}$, for $i = 1, 2$, with $P_i \in \mathbb{R}^{l_{p_i} \times n}$ and positive vectors $\phi_i \in \mathbb{R}^{l_{p_i}}$. Then, $\mathcal{P}_1 \subseteq \mathcal{P}_2$ if and only if there exists a non-negative matrix $Q \in \mathbb{R}^{l_{p_2} \times l_{p_1}}$ such that*

$$\begin{aligned} QP_1 &= P_2, \\ Q\phi_1 &\leq \phi_2. \end{aligned} \tag{2}$$

***Definition** 3. (Robust Positively Invariant Set) A polyhedral set $\mathcal{P}(\phi) \subseteq \mathbb{R}^n$ is said to be Robust Positively Invariant (RPI) for the system $x_{k+1} = f(x_k, d_k)$, $t \geq 0$, $x_k \in \mathbb{R}^n$, $d_k \in \Delta(\psi) \subseteq \mathbb{R}^{n_d}$, where $\Delta(\psi)$ is a compact polyhedral set, if for any initial state $x_0 \in \mathcal{P}(\phi)$, the state trajectory x_k remains bounded inside $\mathcal{P}(\phi), \forall\, k \geq 0$ and $\forall\, d_k \in \Delta(\psi)$.*

2.2 Problem Formulation

Consider a Linear Time-Invariant Discrete-Time (LTID) system given by

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{3}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^m$ the control input vector, and $y_k \in \mathbb{R}^p$, with $p \leq m$, the measurement vector. The system matrices (A, B, C) are of suitable dimensions, with (A, B) controllable, (C, A) observable, and $rank \begin{bmatrix} A - I_n & B \\ C & 0 \end{bmatrix} = n + p = n_{cl}$ (i.e., the system is free from transmission zeros at one) (Blanchini and Miani, 2015).

***Remark** 1. For the sake of notation clarity, in the sequel, the dependency of x, y, u from k is omitted, and $x^+ = x_{k+1}$.*

The state and input vectors are subject to the following state and input constraints, represented respectively by the polyhedrons

$$x \in \mathcal{X} = \{x : Xx \leq \mathbf{1}_{l_x}\},\ X \in \mathbb{R}^{l_x \times n}, \tag{4}$$

$$u \in \mathcal{U} = \{u : Uu \leq \mathbf{1}_{l_u}\},\ U \in \mathbb{R}^{l_u \times m}. \tag{5}$$

For tracking purposes, we also assume that y must track a set-point reference signal $r \in \mathbb{R}^p$, where r is bounded in an asymmetric hyperrectangle described by the set

$$\mathcal{R}(\rho) = \{r : Rr \leq \rho\},\ R = \begin{bmatrix} I_p \\ -I_p \end{bmatrix} \in \mathbb{R}^{2p \times 2p}, \tag{6}$$

where $\rho = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix} \in \mathbb{R}^{2p}$ and $\rho_i = [\rho_{i1} \ \cdots \ \rho_{ip}]^T > 0, i = 1, 2$.

Finally, the tracking controller's structure is as follows

$$u = Ky + K_I x_I + K_r r, \tag{7}$$

where $K, K_I, K_r \in \mathbb{R}^{m \times p}$, $x_{I,k} = \sum_{j=0}^{k-1} e_j \in \mathbb{R}^p$, and $e = r - y$.

***Remark** 2. Note that Ky and $K_I x_I$ define a Proportional and Integral effect, respectively, while $K_r r$ is a feedforward term used to improve the set-point response, see (Åström et al., 2006, Chapter 5).*

Consequently, according to the IMP (Fadali and Visioli, 2013; Chen, 2014, Section 9.3), any stabilizing controller *having the structure of* (7)*, guarantees asymptotic offset-free set-point tracking. However, since the considered system is subject to state and input constraints, the validity of such a result may be restricted to a bounded state space region where the constraints are inactive. Note that if the entire state vector can be measured (i.e., $y = x$), then the controller* (7) *has a state feedback term.*

The problem of interest is as follows.

***Problem** 1. (Constrained Set-Point Tracking (CSPT)) Consider the constrained plant's model* (3)-(5)*, the reference constraint* (6) *and the controller's structure* (7)*. Design the control gains (K, K_I, K_r) in* (7)*, the vector ρ in* (6)*, and a RPI set $\mathcal{F} \subset \mathbb{R}^{n+p}$ such that for any initial condition $x_{cl,0} = [x_0^T \; x_{I,0}^T]^T \in \mathcal{F}$, the set-point reference r is asymptotically tracked and the constraints* (4)-(5) *are fulfilled.*

3. PROPOSED CSPT SOLUTION

This section develops a solution for the CSPT Problem 1 using polyhedral robust positive invariance arguments. In particular, first, the closed-loop dynamics of (3) under (7) are considered, and the associated constraints are explicit. Then, by resorting to proper set inclusion conditions and the Extended Farkas' Lemma, all the necessary algebraic conditions characterizing the set of admissible controller's parameters, reference's bounds, and RPI sets $\mathcal{F}$ are derived (see Proposition 1). Finally, the resulting optimization problem for control design is presented (see opt. (18)).

The following linear system describes the closed-loop dynamics of (3) under the actions of the controller (7),

$$\begin{bmatrix} x^+ \\ x_I^+ \end{bmatrix} = \underbrace{\begin{bmatrix} A + BKC & BK_I \\ -C & I_p \end{bmatrix}}_{A_{cl}} \underbrace{\begin{bmatrix} x \\ x_I \end{bmatrix}}_{x_{cl}} + \underbrace{\begin{bmatrix} BK_r \\ I_p \end{bmatrix}}_{B_{cl}} r. \qquad (8)$$

Furthermore, from (7) the input constraint (5) is translated into a closed-loop constraint, as follows

$$\begin{bmatrix} x_{cl} \\ r \end{bmatrix} \in \mathcal{U}_{cl} = \left\{ \begin{bmatrix} x_{cl} \\ r \end{bmatrix} : U\,[KC \;\; K_I \;\; K_r] \begin{bmatrix} x_{cl} \\ r \end{bmatrix} \leq \mathbf{1}_{l_u} \right\}. \quad (9)$$

Since fast error-tracking dynamics are desirable, to minimize the magnitude of the vector x_I, impose a further optional constraint. In particular, we allow the possibility of bounding each component of x_I in the asymmetric interval $-\xi_{2j}^{-1} \leq x_{I,i} \leq \xi_{1j}^{-1}$, with $\xi_{ij} > 0$ for $i = 1, 2$ and $j = 1, \ldots, p$ or, equivalently,

$$x_I \in \mathcal{X}_I = \{x_I : X_I x_I \leq \mathbf{1}_{2p}\}, \qquad (10)$$

with $X_I = \begin{bmatrix} X_{I_1} \\ -X_{I_2} \end{bmatrix} \in \mathbb{R}^{2p \times p}$, $X_{I_i} = diag\{\xi_{ij}\} \in \mathbb{R}^{p \times p}$.

***Remark** 3. X_I will later be considered as decision matrix variable of the proposed design methodology (see* (17)*).*

Thus, the set of state constraints acting on the closed-loop system (i.e., (4) and (10)) can be re-written as the following single constraint

$$x_{cl} \in \mathcal{X}_{cl} = \left\{ x_{cl} : X_{cl} x_{cl} \leq \mathbf{1}_{l_{x_{cl}}} \right\}, \qquad (11)$$

with $X_{cl} = \begin{bmatrix} X & 0 \\ 0 & X_I \end{bmatrix} \in \mathbb{R}^{l_{x_{cl}} \times n_{cl}}$, $l_{x_{cl}} = l_x + l_{x_I}$.

Since the IMP is only locally valid for the considered constrained system (see Remark 2), the pursued idea is to characterize an RPI polyhedral set $\mathcal{F}$ for (8) where the state trajectory x_{cl} is confined and constraints (9), (11) are fulfilled for any admissible reference signal.

***Remark** 4. Note that in* (8)*, r can be interpreted as a bounded disturbance. Consequently, the RPI nature of $\mathcal{F}$ can be interpreted from Definition 3 performing the following substitutions: $x^+ = f(x, d) \leftarrow$* (8)*, $\mathcal{P}(\phi) \leftarrow \mathcal{F}$, $d \leftarrow r$, and $\Delta(\psi) \leftarrow \mathcal{R}(\rho)$.*

Let $\mathcal{F}$ be described by the polyhedral set

$$\mathcal{F} = \{x_{cl} : F_{cl} x_{cl} \leq \mathbf{1}_{l_f}\}, \qquad (12)$$

with $F_{cl} \in \mathbb{R}^{l_f \times n_{cl}}$ and $rank(F_{cl}) = n_{cl}$. Thus, it is possible to state Proposition 1 that defines the algebraic conditions under which the controller (7) provides a solution to Problem 1.

***Proposition** 1. Consider the closed-loop system* (8) *and the polyhedral sets* (6), (9) *and* (11)*. Assume that there exist matrices $F \in \mathbb{R}^{l_f \times n}$, $F_I \in \mathbb{R}^{l_f \times p}$, $V_1 \in \mathbb{R}^{n \times l_f}$, $V_2 \in \mathbb{R}^{p \times l_f}$ with $l_f > n_{cl}$, non-negative matrices $H \in \mathbb{R}^{l_f \times l_f}$, $H_r \in \mathbb{R}^{l \times l_r}$, $T_1 \in \mathbb{R}^{l_x \times l_f}$, $T_2 \in \mathbb{R}^{l_{x_I} \times l_f}$, $Q \in \mathbb{R}^{l_u \times l_f}$, $Q_r \in \mathbb{R}^{l_u \times l_r}$, and a scalar $0 \leq \lambda < 1$ satisfying*

$$\begin{aligned} [F \;\; F_I] \begin{bmatrix} A + BKC & BK_I \\ -C & I_p \end{bmatrix} &= H\,[F \;\; F_I]\,, \\ [F \;\; F_I] \begin{bmatrix} BK_r \\ I_p \end{bmatrix} &= H_r R, \\ H\mathbf{1}_l + H_r \rho &\leq \lambda 1_l, \end{aligned} \qquad (13)$$

$$\begin{aligned} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} [F \;\; F_I] &= \begin{bmatrix} X & 0 \\ 0 & X_I \end{bmatrix}, \\ T_1 \mathbf{1}_l &\leq \mathbf{1}_{l_{x_{cl}}}, \\ T_2 \mathbf{1}_l &\leq \mathbf{1}_{l_{x_{cl}}}, \end{aligned} \qquad (14)$$

$$\begin{aligned} Q\,[F \;\; F_I] &= U\,[KC \;\; K_I]\,, \\ Q_r R &= U K_r, \\ Q\mathbf{1}_l + Q_r \rho &\leq \mathbf{1}_{l_u}, \end{aligned} \qquad (15)$$

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} [F \;\; F_I] = \mathbb{I}_{n_{cl}}. \qquad (16)$$

Then, the polyhedral set $\mathcal{F}$, defined by (12) *with $F_{cl} = [F \;\; F_I]$, is robust positively invariant and such that $\mathcal{F} \subseteq \mathcal{X}_{cl}$ and $[KC \;\; K_I]\,\mathcal{F} \oplus K_r \mathcal{R}(\rho) \subseteq \mathcal{U}$, where $\oplus$ denotes the Minkowski set sum operator.* Therefore, for any initial condition $x_{cl,0} = [x_0^T \; x_{I,0}^T]^T \in \mathcal{F}$ the output y asymptotically tracks any set-point reference $r \in \mathcal{R}(\rho)$, with corresponding closed-loop trajectories fulfilling the prescribed constraints.

Proof: The proof is adapted from Santos et al. (2023) and reported in Appending A. □

Note that for the RPI set $\mathcal{F}$, we assume that $l_f > n_{cl}$. Then the existence of V that verifies (16) is equivalent to $rank(F_{cl}) = n_{cl}$. Therefore, the first equality in (13) can be interpreted as a generalized similarity transformation that relates the spectral sets of A_{cl} and H, given by $\sigma(A_{cl}) = \{\mu_i, i = 1, \ldots, n\}$ and $\sigma(H) = \{\omega_j, j = 1, \ldots, l_f\}$, respectively. Thus, in the case $rank(F_{cl}) = n_{cl} < l_f$, we have $\sigma(A_{cl}) \subseteq \sigma(H)$. If $0 \leq \lambda < 1$, the inequality in (13) implies that the non-negative matrix H is Schur because the elements of its spectrum are $|\omega_i| \leq \bar{\omega} \leq \lambda$, where $\bar{\omega}$ is necessarily a non-negative real eigenvalue belonging to $\sigma(H)$ (Hennet, 1995; Brião et al., 2021). Hence, we can conclude that A_{cl} is also Schur and $|\mu_i| \leq \lambda$.

Note that under the conditions (13)-(16), for any reference signal $r \in \mathcal{R}(\rho)$ and for all $x_{cl,0} = [x_0^T \ x_{I,0}^T]^T \in \mathcal{F}$, the closed-loop state trajectory remains inside $\mathcal{F}$ while fulfilling all the prescribed state and input constraints. Consequently, the system evolves in a domain where the constraints are inactive. Thus, the closed-loop dynamics is uniquely determined by the unconstrained linear model (8), whose state matrix A_{cl} is Schur stable. Hence, the IMP is locally valid for any $r \in \mathcal{R}(\rho)$ and for all $x_{cl,0} = [x_0^T \ x_{I,0}^T]^T \in \mathcal{F}$.

3.1 Bilinear programming design approach

From Proposition 1, the algebraic relations (13)-(16) define the constraints under which the controller provides a solution to Problem 1, where the set of decision variables for the design of the controller (7) is given by

$$\lambda(\cdot) = (K, K_I, K_r, F_{cl}, H, H_r, T, Q, Q_r, V, X_I, \lambda, \rho). \quad (17)$$

There hence, the resulting bilinear optimization problem formulates as

$$\begin{aligned} \underset{\lambda(\cdot)}{\text{maximize}} \quad & \Phi(\cdot), \\ \text{subject to} \quad & (13) - (16), \\ & f_\ell(\cdot) \leq \varphi_\ell, \ \ell = 1, \ldots, \bar{\ell}, \end{aligned} \quad (18)$$

where $\Phi(\cdot)$ is the cost function and $f_\ell(\cdot) \leq \varphi_\ell$ are $\bar{\ell}$ auxiliary constraints instrumental to imposing limits over all the non-bounded decision variables.

Concerning the cost function $\Phi(\cdot)$, the choice is not unique and depends on the designer's objectives. Two possible options are:

i) $\Phi(\cdot) = \Phi_1 = \|\rho\|_1$. This choice allows us to maximize the hyperrectangle $\mathcal{R}(\rho)$ of all admissible reference signals.
ii) $\Phi(\cdot) = \Phi_2 = trace(X_{I_1} + X_{I_2})$. This choice allows us to minimize the limits of the admissible integral errors $\xi_{ij}^{-1} > 0$, for $i = 1, 2$ and $j = 1, ..., p$ (see (10)).

It is essential to note that the opt. (18) is bilinear because it involves multiplication between decision (matrices and vector) variables. Therefore, (18) can be solved by employing nonlinear optimization techniques. In this regard, the extra constraints $f_\ell(\cdot) \leq \varphi_\ell$ serve the purpose of reducing the search space and improving the numerical performance of the used nonlinear optimizer.

***Remark** 5. The difference between the conditions of Proposition 1 and the continuous-time counterpart (Proposition 1 in Santos et al. (2023)) is that* (13) *should be changed by the algebraic relations that describe the robust positive invariance of the set $\mathcal{F}$ for continuous-time systems. The other ones, representing the set inclusions and the full-column rank condition upon F_{cl}, remain the same. Thus, a similar bilinear optimization problem* (18) *applies for continuous-time systems where, in particular, H and λ should be a Metzler-type matrix and a negative real scalar, respectively. Also, if* (3) *corresponds to a continuous-time model's discretization with sampling-period T_s sec., then the continuous and discrete-time integral-error obeys*

$$x_I(t) = \int_0^{t=kT_s} e(\tau)d\tau \approx T_s x_{I,k}. \quad (19)$$

One possible way to solve the bilinear optimization (18) is to resort to the nonlinear state-of-the-art solver KNITRO (Byrd et al., 2006), which has already been successfully employed for similar problems in Santos et al. (2021); Brião et al. (2021); Ernesto et al. (2021); Martins et al. (2020). Thus, one can find the bounds of the decision space using insights about the plant's constraint limits and a trial-and-error approach. For further discussions about KNITRO and its use, see (Brião et al., 2021, Section 4.2).

Table 1 summarizes the number of variables and constraints characterizing (18). Notice that the proposed solution's complexity increases with the plant's dimensions (i.e., system model, state and input constraints, reference set) and RPI set $\mathcal{F}$ complexity, i.e., the number of inequalities l_f used to describe $\mathcal{F}$ (Santos et al., 2023).

Table 1. Variables, equalities and inequalities in (18)

# of variables	$mp^3 + l_f(n_{cl} + l_f + l_r + l_{x_{cl}} + l_u)$ $+2p(l_u + p + 1) + n_{cl}^2 + 1$
# of equalities	$n_{cl}(l_f + l_{x_{cl}} + l_u + n_{cl}) + 2p(l_f + l_u)$
# of inequalities	$l_f + l_{x_{cl}} + l_u$

4. NUMERICAL EXAMPLE

In this section, the proposed tracking controller design's effectiveness is validated through simulation results obtained for a linearized model of a two-tank system. In this example, the state vector cannot be entirely measured, and the controller's performance is evaluated for the two proposed cost functions.

Moreover, the optimization (18) has been solved using KNITRO (Byrd et al., 2006), where we bounded the optimization variables (element by element) similarly as in Santos et al. (2023) for comparative purposes with the continuous-time design:

$$\begin{aligned} H, H_r, T_1, T_2, Q, Q_r \text{ in } & \left[0, 10^2\right], \\ F_{cl}, K, K_I, K_r \text{ in } & \left[-10^2, 10^2\right], \\ V \text{ in } & \left[-10^3, 10^3\right]. \end{aligned}$$

***Example** 1.* Consider the discrete-time system obtained from the ZOH-discretization of the continuous-time model used in Santos et al. (2023) (adapted from Ferramosca et al. (2011)), with sampling period $T_s = 1$ second:

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0.9970 & 0.0182 \\ 0 & 0.9814 \end{bmatrix} x_k + \begin{bmatrix} 6.6583 \\ 9.9070 \end{bmatrix} u_k, \\ y_k &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_k, \end{aligned} \quad (20)$$

where the state and input constraints are: $-0.38 \leq x_1 \leq 0.68$, $-0.35 \leq x_2 \leq 0.65$, and $-2 \leq u \leq 2$.

We have solved the optimization problem (18) for both $\Phi(\cdot) = \Phi_1$ and $\Phi(\cdot) = \Phi_2$ using $\lambda = 0.9999$ and $l_f = 9$. Table 2 summarizes and compares the design results for both objective functions. In particular, ρ defines the bounds of the set of admissible reference signals $\mathcal{R}(\rho)$, X_I is the shaping matrix of the set constraining the integral error, and the gains K, K_I and K_r define the control law (7). As expected, for the cost function Φ_1, the bounds for the admissible reference signals $\mathcal{R}$ are bigger than the ones obtained for Φ_2. On the other hand, by using Φ_2, it is possible to obtain a smaller integral error and faster set-point tracking. Indeed, for Φ_1, $x_I \in [-14,\ 17]$, while for Φ_2, $x_I \in [-11,\ 11]$. However, the cost to pay is a reduced size for the set of admissible reference $\mathcal{R}(\rho)$.

Table 2. Design results using (18): Φ_1 vs Φ_2

Φ_i	ρ	X_I	$[K\ K_I\ K_r]^T$
1	$\begin{bmatrix}0.6662\\0.3587\end{bmatrix}$	$\begin{bmatrix}0.0574\\-0.0682\end{bmatrix}$	$[-0.1544\ \ 0.0046\ \ 0.0970]^T$
2	$\begin{bmatrix}0.4600\\0.3300\end{bmatrix}$	$\begin{bmatrix}0.0900\\-0.0900\end{bmatrix}$	$[-0.0221\ \ 0.0007\ \ 0.0081]^T$

The resulting RPI set $\mathcal{F}$ is depicted in Figure 1 with two closed-loop trajectories obtained using the tracking controller associated to $\Phi(\cdot) = \Phi_2$. They have been obtained starting from a zero initial condition and considering the two extreme points of the admissible reference signals, i.e., $r^{up} = \rho_1 = 0.4600.$ and $r^{down} = -\rho_2 = -0.3300$. The corresponding equilibrium states are, respectively, $x_{cl}^{up} = [0.4600\ 0.4487\ 10.5511]^T$ and $x_{cl}^{down} = [-0.3300\ -0.3219\ -7.5693]^T$. The obtained trajectories confirm that the designed tracking controller allows the plant to asymptotically track the assigned set-point reference signals while ensuring that the state trajectory remains confined in the constraint-admissible RPI set $\mathcal{F}$.

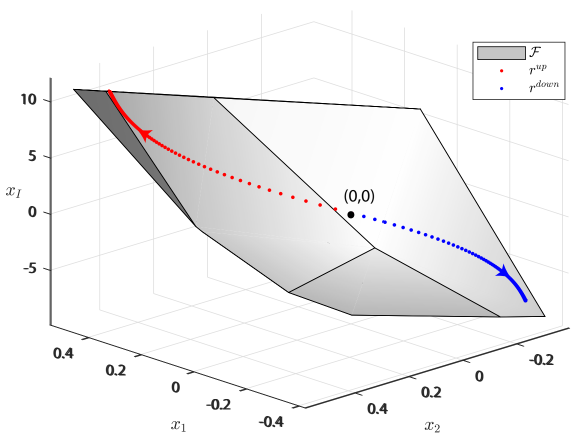

Figure 1. RPI set $\mathcal{F}$ and state trajectories for $r^{up} = \rho_1$ ($\cdot\cdot$) and $r^{down} = -\rho_2$ ($\cdot\cdot$).

Finally, it is worth mentioning that the reported numerical results of ρ and X_I agree with the corresponding results obtained in continuous-time (Santos et al. (2023), Example 1). For comparative purposes, Figure 2 depicts both the continuous and discrete-time evolution of the output, $y(t)$ and y_k, and integral error, $x_I(t)$, and its discrete-time-approximation $x_{I,k}$ (see Remark 5, eq. (19)). We observe that the continuous and discrete-time designs yield similar temporal behaviors for both reference tracking and integral-error states.

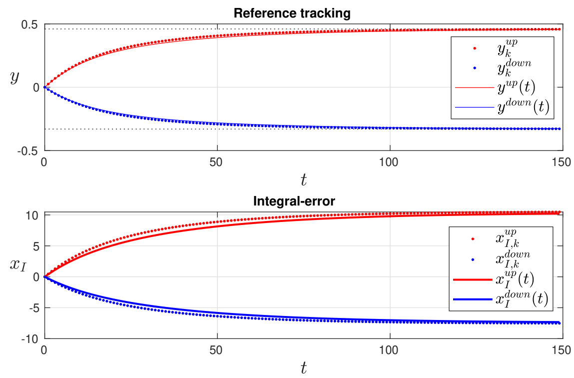

Figure 2. *Top:* Output and Reference tracking; *Botton:* Integral-error. In both figures, dotted lines refer to discrete-time evolution, and the superscripts *up* and *down* correspond to the trajectories obtained for $r^{up} = \rho_1$ and $r^{down} = \rho_2$, respectively.

5. CONCLUSION

We have proposed the discrete-time counterpart of the PI-like set-point tracking controller recently presented for linear continuous-time systems subject to polyhedral state and control constraints (Santos et al., 2023). The considered solution leveraged robust positive invariance arguments to define the algebraic conditions under which the proposed controller ensures set-point tracking and constraint fulfillment. A peculiar design's feature consists of a single bilinear programming problem capable of simultaneously computing the controller parameters, the set of admissible reference signals, and the controller's domain of attraction. The numerical example illustrated and compared the proposed design solutions with the continuous-time counterpart results.

Future works will be devoted to extending the proposed approach to deal with disturbances and more complex models for the plant and reference signal, and allowing the control saturation to obtain larger domains of attraction (see (Martins et al., 2020)).

APPENDIX A

Proof of Proposition 1:

1) The existence of the non-negative matrices H and H_r, and the scalar $0 \le \lambda < 1$ verifying the conditions (13) are necessary and sufficient algebraic conditions for the robust positive invariance of the set $\mathcal{F}$, which is the equivalent of imposing the one step admissibility condition $A_{cl}\mathcal{F} \oplus B_{cl}\mathcal{R}(\rho) \subseteq \mathcal{F}$ (Lucia et al., 2023; Blanchini and Miani, 2015).

2) To guarantee that the system will stay within the closed loop state constraints, we impose the inclusion $\mathcal{F} \subseteq \mathcal{X}_{cl}$, which, by applying the Extended Farkas' Lemma 1, is equivalent to the existence of the non-negative matrix $T = \left[T_1'\ T_2'\right]'$, that satisfies the relation (14).

3) Likewise, applying Lemma 1, the existence of non-negative matrices Q and Q_r verifying (15), represents the inclusion $[KC\ \ K_I]\,\mathcal{F} \oplus K_r\mathcal{R}(\rho) \subseteq \mathcal{U}$ or, equivalently,

$$[\,Q\ Q_r\,]\begin{bmatrix} F & 0 \\ 0 & R \end{bmatrix} = U\,[\,KC\ K_I\ K_r\,]\ ,\ [Q\ Q_r]\begin{bmatrix} \mathbf{1}_l \\ \rho \end{bmatrix} \le \mathbf{1}_{l_u}\,.$$

4) Finally, condition (16) is equivalent to imposing $rank(F_{cl}) = n_{cl}$ and, possibly, that $\mathcal{F}$ is compact.

Then, the proof is completed by using local stability arguments and the IMP.

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